Measures of risk aversion

Absolute Risk Aversion

The higher the curvature of $u(c)$, the higher the risk aversion. However, since expected utility functions are not uniquely defined (are defined only up to affine transformations), a measure that stays constant with respect to these transformations is needed. One such measure is the Arrow-Pratt measure of absolute risk-aversion (ARA), after the economists Kenneth Arrow and John W. Pratt, also known as the coefficient of absolute risk aversion, defined as

$$A(c) = -\frac{u''(c)}{u'(c)}.$$

The following expressions relate to this term:

- **Exponential utility** of the form $u(c) = 1 - e^{-\alpha c}$ is unique in exhibiting constant absolute risk aversion (CARA): $A(c) = \alpha$ is constant with respect to $c$.

- **Hyperbolic absolute risk aversion** (HARA) is the most general class of utility functions that are usually used in practice (specifically, CRRA (constant relative risk aversion, see below), CARA (constant absolute risk aversion), and quadratic utility all exhibit HARA and are often used because of their mathematical tractability). A utility function exhibits HARA if its absolute risk aversion is a hyperbolic function, namely

  $$A(c) = -\frac{u''(c)}{u'(c)} = \frac{1}{ac + b}.$$

  The solution to this differential equation (omitting additive and multiplicative constant constant terms, which do not affect the behavior implied by the utility function) is:

  $$u(c) = \frac{(c - c_s)^{1-R}}{1 - R}$$

  where $R = 1 / a$ and $c_s = -b / a$. Note that when $a = 0$, this is CARA, as $A(c) = 1 / b = \text{const}$, and when $b = 0$, this is CRRA (see below), as $cA(c) = 1 / a = \text{const}$.

- **Decreasing/increasing absolute risk aversion** (DARA/IARA) if $A(c)$ is decreasing/increasing. An example of a DARA utility function is $u(c) = \log(c)$, with $A(c) = 1 / c$, while $u(c) = c - \alpha c^2$, $\alpha > 0$, with $A(c) = 2\alpha / (1 - 2\alpha c)$ would represent a quadratic utility function exhibiting IARA.

- Experimental and empirical evidence is mostly consistent with decreasing absolute risk aversion.$^{[4]}$

- Contrary to what several empirical studies have assumed, wealth is not a good proxy for risk aversion when studying risk sharing in a principal-agent setting.
Although is monotonic in wealth under either DARA or IARA and constant in wealth under CARA, tests of contractual risk sharing relying on wealth as a proxy for absolute risk aversion are usually not identified. [5]

**RELATIVE RISK AVERSION**

The Arrow-Pratt measure of relative risk-aversion (RRA) or coefficient of relative risk aversion is defined as

\[ R(c) = cA(c) = \frac{-cu''(c)}{u'(c)}. \]

Like for absolute risk aversion, the corresponding terms constant relative risk aversion (CRRA) and decreasing/increasing relative risk aversion (DRRA/IRRA) are used. This measure has the advantage that it is still a valid measure of risk aversion, even if the utility function changes from risk-averse to risk-loving as \( c \) varies, i.e. utility is not strictly convex/concave over all \( c \). A constant RRA implies a decreasing ARA, but the reverse is not always true. As a specific example, the expected utility function \( u(c) = \log(c) \) implies RRA = 1.

In intertemporal choice problems, the elasticity of intertemporal substitution is often unable to be disentangled from the coefficient of relative risk aversion. The isoelastic utility function

\[ u(c) = \frac{c^{1-\rho}}{1-\rho} \]

exhibits constant relative risk aversion with \( R(c) = \rho \) and the elasticity of intertemporal substitution \( \varepsilon_{u(c)} = 1/\rho \).

When \( \rho = 1 \) and one is subtracted in the numerator (facilitating the use of l'Hôpital's rule), this simplifies to the case of log utility, and the income effect and substitution effect on saving exactly offset.

**IMPLICATIONS OF INCREASING/DECREASING ABSOLUTE AND RELATIVE RISK AVERSION**

The most straightforward implications of increasing or decreasing absolute or relative risk aversion, and the ones that motivate a focus on these concepts, occur in the context of forming a portfolio with one risky asset and one risk-free asset. If the person experiences an increase in wealth, he/she will choose to increase (or keep unchanged, or decrease) the number of dollars of the risky asset held in the portfolio if absolute risk aversion is decreasing (or constant, or increasing). Thus economists avoid using utility functions, such as the quadratic, which exhibit increasing absolute risk aversion, because they have an unrealistic behavioral implication.

Similarly, if the person experiences an increase in wealth, he/she will choose to increase (or keep unchanged, or decrease) the fraction of the portfolio held in the risky asset if relative risk aversion is decreasing (or constant, or increasing).
**Instrumental variables (IV)**

In statistics, econometrics, epidemiology and related disciplines, the method of *instrumental variables* (IV) is used to estimate causal relationships when controlled experiments are not feasible.

Statistically, IV methods allow consistent estimation when the explanatory variables (covariates) are correlated with the error terms. Such correlation may occur when the dependent variable causes at least one of the covariates ("reverse" causation), when there are relevant explanatory variables which are omitted from the model, or when the covariates are subject to measurement error. In this situation, ordinary linear regression generally produces biased and inconsistent estimates. However, if an *instrument* is available, consistent estimates may still be obtained. An instrument is a variable that does not itself belong in the explanatory equation and is correlated with the endogenous explanatory variables, conditional on the other covariates. In linear models, there are two main requirements for using an IV:

- The instrument must be correlated with the endogenous explanatory variables, conditional on the other covariates.
- The instrument cannot be correlated with the error term in the explanatory equation, that is, the instrument cannot suffer from the same problem as the original predicting variable.

**Regression analysis**

In statistics, *regression analysis* includes any techniques for modeling and analyzing several variables, when the focus is on the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps us understand how the typical value of the dependent variable changes when any one of the independent variables is varied, while the other independent variables are held fixed.

**Affine transformation**

In geometry, an *affine transformation* or *affine map* or an *affinity* (from the Latin, *affinis*, "connected with") between two vector spaces (strictly speaking, two affine spaces) consists of a linear transformation followed by a translation:

\[ x \mapsto Ax + b. \]
In the finite-dimensional case each affine transformation is given by a matrix $A$ and a vector $b$, satisfying certain properties described below. Geometrically, an affine transformation in Euclidean space is one that preserves:

1. The collinearity relation between points; i.e., the points which lie on a line continue to be collinear after the transformation
2. Ratios of distances along a line; i.e., for distinct collinear points $p_1, p_2, p_3$, the ratio $|p_2 - p_1| / |p_3 - p_2|$ is preserved

In general, an affine transformation is composed of linear transformations (rotation, scaling or shear) and a translation (or "shift"). Several linear transformations can be combined into a single one, so that the general formula given above is still applicable.

In the one-dimensional case, $A$ and $b$ are called, respectively, slope and intercept.

**Method of Least squares**

The method of least squares is a standard approach to the approximate solution of over-determined systems, i.e. sets of equations in which there are more equations than unknowns. "Least squares" means that the overall solution minimizes the sum of the squares of the errors made in solving every single equation. The most important application is in data fitting. The best fit in the least-squares sense minimizes the sum of squared residuals, a residual being the difference between an observed value and the fitted value provided by a model.

The result of fitting a set of data points with a quadratic function.